Lesson 20. Optimization with Equality Constraints, cont.

Example 1. Suppose that you are interested in dividing your savings between three mutual funds with expected returns of 10%, 10% and 15%, respectively. You want to minimize risk while achieving an expected return of 12%. To measure risk, use the *variance* of the return on investment: when a fraction x of your savings is invested in Fund 1, y in Fund 2, and z in Fund 3, the variance of the return is

$$v(x, y, z) = 400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz$$

a. Consider the equality constraints below. Why do these constraints make sense for this problem?

expected return of portfolio must be 12% $\longrightarrow$ $1.10x + 1.10y + 1.15z = 1.12$

$x + y + z = 1$ $\longleftarrow$ fractions of savings must add up to 1

b. Find the local optima of the variance of the return v, subject to the equality constraints given in part a.

c. How much should you invest in the three mutual funds?

b. $$L(\lambda_1, \lambda_2, x, y, z) = 400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz - \lambda_1[1.10x + 1.10y + 1.15z - 1.12] - \lambda_2[x + y + z - 1]$$

$$\nabla L(\lambda_1, \lambda_2, x, y, z) = \begin{bmatrix} -(1.10x + 1.10y + 1.15z - 1.12) \\ -(x + y + z - 1) \\ 800x + 200y - 1.10\lambda_1 - \lambda_2 \\ 200x + 1600y + 400z - 1.10\lambda_1 - \lambda_2 \\ 400y + 3200z - 1.15\lambda_1 - \lambda_2 \end{bmatrix}$$

Solve for CCPs:

$$\begin{aligned} 1.10x + 1.10y + 1.15z &= 1.12 \\ x + y + z &= 1 \\ -1.1\lambda_1 - \lambda_2 + 800x + 200y &= 0 \\ -1.1\lambda_1 - \lambda_2 + 200x + 1600y + 400z &= 0 \\ -1.15\lambda_1 - \lambda_2 + 400y + 3200z &= 0 \end{aligned}$$

$\Rightarrow$ 1 CCP: $(18000, -19380, 0.5, 0.1, 0.4)$

$\uparrow$

e.g. form augmented matrix, find RREF

$$H_L(\lambda_1, \lambda_2, x, y, z) = \underbrace{\begin{bmatrix} 0 & 0 & -1.1 & -1.1 & -1.15 \\ 0 & 0 & -1 & -1 & -1 \\ -1.1 & -1 & 800 & 200 & 0 \\ -1.1 & -1 & 200 & 1600 & 400 \\ -1.15 & -1 & 0 & 400 & 3200 \end{bmatrix}}_{*}$$

2nd deriv. test:

$$H_L(18000, -19380, 0.5, 0.1, 0.4) = *$$

$\Rightarrow k = 2 \qquad 2k+1 = 5 \qquad d_5 = |H_L(18000, -19380, 0.5, 0.1, 0.4)| = 5$

$\quad\; n = 3 \qquad n + k = 5 \qquad (-1)^k d_5 = 5 > 0$

$\Rightarrow$ v has a constrained local minimum at $(0.5, 0.1, 0.4)$

c. (Assuming a local optimum is good enough)
You should invest 50% in Fund 1, 10% in Fund 2, and 40% in Fund 3 to achieve an expected return of 12% at minimum risk (variance).